The Decay of Multiqudit Entanglement

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We investigate the decay of entanglement of a generalized N-qudit GHZ state with each qudit passing through independently in a quantum noisy channel. By studying the time at which the entanglement completely vanishes and the time at which the entanglement becomes arbitrarily small, we try to find how the robustness of entanglement is influenced by dimension d and the number of particles N.

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I. INTRODUCTION

Quantum entanglement, as the most non-classical phenomenon in quantum mechanics, lies in the central position of quantum information theory and has been identified as a key resource in many applications such as quantum teleportation, quantum key distribution and quantum computation [1], see Ref.[12] for a review. For large-scale quantum information processing, multiparticle entanglement is indispensable. Therefore the understanding of the dynamical property of multiparticle entanglement in realistic environment is of fundamentally importance.

Due to the interaction with environment, multiparticle entanglement decays inevitably. In the past several years, there are many excellent papers concerning with the robustness of multiparticle entanglement under the influence of environment [2, 4, 5, 6, 7, 8, 10, 11]. It was shown that for a N-qubit GHZ state (throughout N is even for simplicity) $\frac{1}{\sqrt{2}}(|0\rangle^{\otimes N}+|1\rangle^{\otimes N})$ with each qubit exposed to a depolarizing channel independently, the bipartite entanglement of $(N-n)|n$ bipartition disappears in finite time which is named as entanglement sudden death (ESD) in Refs. [2, 6, 10, 11]. The entanglement which first disappears is that corresponding to $1|(N-1)$ bipartition and the time at which this entanglement disappears decreases with N. The entanglement which last disappears is that corresponding to $N/2|N/2$ bipartition and the time at which this entanglement disappears grows with N [2, 4]. Aolita *et al.* [8] proposed that the time at which the entanglement becomes arbitrarily small is a better quantity characterizing the robustness of entanglement than the ESD time and they found that this time is inversely proportional to N for N-qubit generalized GHZ state $|\Psi_2\rangle=\alpha|0\rangle^{\otimes N}+\beta|1\rangle^{\otimes N}$.

It is understandable that the robustness of multiparticle entanglement not only depends on the number of particles but also on the dimensionality of each particle's Hilbert space. And for different quantum channels, the behavior of the multiparticle entanglement will also be different. However, it is necessary to have a fixed example to see how all of these results can be obtained. In this article, we will study those problems. We suppose each particle is in a d-dimensional Hilbert space and it is called generally a qudit. We consider a simple case of a generalized N-qudit GHZ state with each qudit interacts with a quantum channel independently and find how does the robustness of entanglement change with the increase of d or N through studying the time evolution of the bipartite entanglement of $(N-n)|n$ bipartition. We use the depolarizing channel and the phase damping channel as the quantum channels.

II. QUANTUM CHANNELS

First we would like to introduce two $d\times d$ matrices extremely useful in constructing our quantum channels. They are defined as $X|i\rangle=|i+1\rangle(\text{mod } d)$ and $Z|i\rangle=\omega^i|i\rangle$, where $i=0,...,d-1$ and $\omega=\exp(2\pi \mathrm{i}/d)$. One can easily see that when $d=2$, they are just Pauli-sigma x and Pauli-sigma z matrix respectively. Now we will construct three operator transformations through X and Z:

$$\mathcal{E}_1(\mathcal{A})=(1-p)\mathcal{A}+\frac{p}{d^2}\sum_{ij=0}^{d-1}X^iZ^j\mathcal{A}Z^{\dagger j}X^{\dagger i},$$

$$\mathcal{E}_2(\mathcal{A})=(1-p)\mathcal{A}+\frac{p}{d}\sum_{i=0}^{d-1}Z^i\mathcal{A}Z^{\dagger i}, \tag{1}$$

where $p\in[0,1]$ and $\mathcal{A}$ is an arbitrary operator on d-dimensional Hilbert spaces. When $\mathcal{A}$ is a density matrix, we can see $\mathcal{E}_i, i=1,2$ as three quantum channels. Through calculation, it can be shown that for an input density matrix ρ, $\mathcal{E}_1(\rho)=(1-p)\rho+\frac{p}{d}\mathbf{1}$ and $\mathcal{E}_2(\rho)=(1-p)\rho+p\sum_{k=0}^{d-1}\rho_{kk}|k\rangle\langle k|$, where $\mathbf{1}$ is a $d\times d$ identity matrix. It's obvious to see that in fact $\mathcal{E}_1$ is a depolarizing channel and $\mathcal{E}_2$ is a phase damping channel.

III. THE EVOLUTION OF ENTANGLEMENT

A generalized N-qudit GHZ state can be written as $|\Psi_d\rangle=\sum_{i=0}^{d-1}\alpha_i|i\rangle^{\otimes N}$, where α_i is a complex number

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and $\sum_{i=0}^{d-1}|\alpha_i|^2=1$. Here we want to show how does the bipartite entanglement corresponding to $(N-n)|n$ bipartition of it evolve under the influence of the three quantum channels constructed above. In this article we adopt negativity as the measure of entanglement [3].

When each qudit is exposed to a depolarizing channel, we can calculate $\left(\bigotimes_{i=1}^{N}\mathcal{E}_{1,i}\right)(|\Psi_d\rangle\langle\Psi_d|)$ and denote it as $\rho_1(p)\equiv\sum_{i=0}^{d-1}\sum_{k=0}^{N}\sum_{\mathcal{P}}|\alpha_i|^2\left(\frac{p}{d}\right)^k\left(1-\frac{d-1}{d}p\right)^{N-k}\times\mathcal{P}\Big[(|i\rangle\langle i|)^{\otimes(N-k)}\otimes\Big(\sum_{j=0,j\neq i}^{d-1}|j\rangle\langle j|\Big)^{\otimes k}\Big]+(1-p)^N\sum_{i=0}^{d-1}\sum_{j=0,j\neq i}^{d-1}\alpha_i\alpha_j^*(|i\rangle\langle j|)^{\otimes N}$, where $\mathcal{P}$ means all possible permutations. Partial transposing the part of n particles of $\rho_1(p)$ and noting that the first term of $\rho_1(p)$ is diagonal so it will not be changed after partial transpose, we have $\rho_1(p)^{\Gamma_n}=\sum_{i=0}^{d-1}\sum_{k=0}^{N}\sum_{\mathcal{P}}|\alpha_i|^2\left(\frac{p}{d}\right)^k\left(1-\frac{d-1}{d}p\right)^{N-k}\times\mathcal{P}\Big[(|i\rangle\langle i|)^{\otimes(N-k)}\otimes\Big(\sum_{j=0,j\neq i}^{d-1}|j\rangle\langle j|\Big)^{\otimes k}\Big]+(1-p)^N\sum_{i=0}^{d-1}\sum_{j=0,j\neq i}^{d-1}\alpha_i\alpha_j^*(|i\rangle\langle j|)^{\otimes(N-n)}\otimes(|j\rangle\langle i|)^{\otimes n}$. There are $\frac{d(d-1)}{2}$ eigenvalues μ of $\rho_1(p)^{\Gamma_n}$ that can be negative and are determined by the smaller eigenvalues of the following $\frac{d(d-1)}{2}$ 2×2 matrices $\begin{pmatrix}\lambda_n^{ij} & \alpha_i\alpha_j^*(1-p)^N\\ \alpha_i^*\alpha_j(1-p)^N & \lambda_{N-n}^{ij}\end{pmatrix}$, where $i<j$ and $\lambda_n^{ij}=|\alpha_i|^2\left(\frac{p}{d}\right)^n\left(1-\frac{d-1}{d}p\right)^{N-n}+|\alpha_j|^2\left(\frac{p}{d}\right)^{N-n}\left(1-\frac{d-1}{d}p\right)^n$. One can easily derive $\mu_n^{ij}=\xi_n^{ij}-\sqrt{(\xi_n^{ij})^2-\eta_n^{ij}}$, where $\xi_n^{ij}=\frac{1}{2}(\lambda_n^{ij}+\lambda_{N-n}^{ij})$ and $\eta_n^{ij}=\lambda_n^{ij}\lambda_{N-n}^{ij}-|\alpha_i\alpha_j|^2(1-p)^{2N}$. Then we can define $\mathcal{N}_n^{ij}=\max\{-\mu_n^{ij},0\}$ and the negativity can be obtained by summation $\mathcal{N}_n(\rho_1(p))=\sum_{i<j=0}^{d-1}\mathcal{N}_n^{ij}$.

Similar to the case of qubit [8], $\mathcal{N}_1^{ij}\leq\mathcal{N}_2^{ij}\leq...\leq\mathcal{N}_{N/2}^{ij}$ for a given pair of α_i and α_j, which straightforwardly leads to $\mathcal{N}_1\leq\mathcal{N}_2\leq...\leq\mathcal{N}_{N/2}$. So the bipartite entanglement corresponding to the most balanced partition still disappears last whereas the one corresponding to the least balanced partition disappears first. Let $p_n=\max_{i<j}p_n^{ij}$, where p_n^{ij} is the solution of the equation $\mu_n^{ij}=0$, then $\mathcal{N}_n(\rho_1(p_n))=0$. Now we investigate dynamical property of $\mathcal{N}_{N/2}$, which disappears last. First we want to know when does it completely vanish. By solving $\mu_{N/2}^{ij}=0$, it's easy to find that

$$p_{N/2}^{ij}=\frac{2|\alpha_i\alpha_j|^{\frac{2}{N}}d}{2|\alpha_i\alpha_j|^{\frac{2}{N}}d+(|\alpha_i|^2+|\alpha_j|^2)^{\frac{1}{N}}\left\{(|\alpha_i|^2+|\alpha_j|^2)^{\frac{1}{N}}+\sqrt{4|\alpha_i\alpha_j|^{\frac{2}{N}}+(|\alpha_i|^2+|\alpha_j|^2)^{\frac{2}{N}}}\right\}},\qquad(2)$$

which coincides with Eq.(6) of Ref.[8] when $d=2$. After p reaches the value $p_{N/2}=\max_{i<j}p_{N/2}^{ij}$, $\mathcal{N}_{N/2}=0$. It's obvious that $p_{N/2}^{ij}<1$ so $p_{N/2}<1$ and so ESD happens. Before $p=p_{N/2}$, $\mathcal{N}_{N/2}>0$ and $\rho_1(p)$ must be still entangled. Second we want to know when does $\mathcal{N}_{N/2}$ become arbitrarily small, which is practically important because before the entanglement is zero, it can be so small that it's useless as a resource. Like Ref.[8], we suppose ϵ is an arbitrarily small positive number and define a critical probability p_ϵ^{ij} such that $\mu_{N/2}^{ij}(p_\epsilon^{ij})=\epsilon\mu_{N/2}^{ij}(0)$, which leads to an equation

$$(|\alpha_i|^2+|\alpha_j|^2)\Big(\frac{p_\epsilon^{ij}}{d}\Big)^{\frac{N}{2}}\Big(1-\frac{d-1}{d}p_\epsilon^{ij}\Big)^{\frac{N}{2}}-|\alpha_i\alpha_j|(1-p_\epsilon^{ij})^N=-\epsilon|\alpha_i\alpha_j|.\qquad(3)$$

When $p\geq p_\epsilon\equiv\max_{i<j}p_\epsilon^{ij}$, we can think $\mathcal{N}_{N/2}$ is too small to be used as a resource.

As the second case let's consider the situation where each qudit is exposed to a phase damping channel. Similar to the case of depolarizing channel, it can be obtained that $\rho_2(p)\equiv\left(\bigotimes_{i=1}^{N}\mathcal{E}_{2,i}\right)(|\Psi_d\rangle\langle\Psi_d|)=\sum_{i=0}^{d-1}|\alpha_i|^2(|i\rangle\langle i|)^{\otimes N}+(1-p)^N\sum_{i=0}^{d-1}\sum_{j=0,j\neq i}^{d-1}\alpha_i\alpha_j^*(|i\rangle\langle j|)^{\otimes N}$ and the partial transpose of it is $\rho_2(p)^{\Gamma_n}=\sum_{i=0}^{d-1}|\alpha_i|^2(|i\rangle\langle i|)^{\otimes N}+(1-p)^N\sum_{i=0}^{d-1}\sum_{j=0,j\neq i}^{d-1}\alpha_i\alpha_j^*(|i\rangle\langle j|)^{\otimes(N-n)}\otimes(|j\rangle\langle i|)^{\otimes n}$. So it's easy to verify that the negativity $\mathcal{N}_n(\rho_2(p))$ can be expressed as $\sum_{i<j=0}^{d-1}\max\{0,-\nu_n^{ij}\}$, where $\nu_n^{ij}=-|\alpha_i\alpha_j|(1-p)^N$ is independent of n. We note that for any n, $\mathcal{N}_n(\rho_2(p))=0$ only when $p=1$, meaning that no ESD happens for phase damping channel. When the negativity becomes arbitrarily small, namely $\nu_n^{ij}(p_\epsilon^{ij})=\epsilon\nu_n^{ij}(0)$, we have an equation $(1-p_\epsilon^{ij})^N=\epsilon$.

In what follows we will focus our attention on the results of entanglement evolution under the depolarizing channel and phase damping channel to study how does the robustness of entanglement change with N and d.

IV. ROBUSTNESS OF ENTANGLEMENT

First we fix d to study the relation between the entanglement robustness and N. For the depolarizing channel, Eq.(2) tells us that the ESD time of $\mathcal{N}_{N/2}$ grows with N (FIG.1). Noting that $\lim_{N\to\infty}p_{N/2}^{ij}=2d/(2d+1+\sqrt{5})$ which is independent of i and j, we find $p_{N/2}=\max_{i<j}p_{N/2}^{ij}$ becomes closer to this value while N

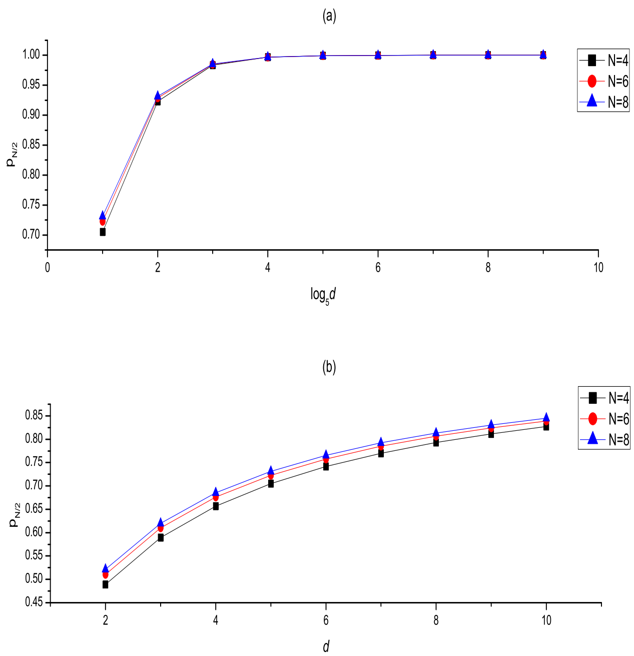

FIG. 1: (color online) For the depolarizing channel, we demonstrate how can $p_{N/2}$ be influenced by both N and d. Here we choose $\alpha_i = 1/\sqrt{d}$ for convenience and $N = 4$ (black cubic), $N = 6$ (red circle) and $N = 8$ (blue triangle) respectively. (a): the behavior of $p_{N/2}$ for large d. (b): the behavior of $p_{N/2}$ for small d. It's easy to see that $p_{N/2}$ grows with N for a fixed value of d saturating to $2d/(2d + 1 + \sqrt{5})$ and it also grows with d and saturates to 1 for any fixed N.

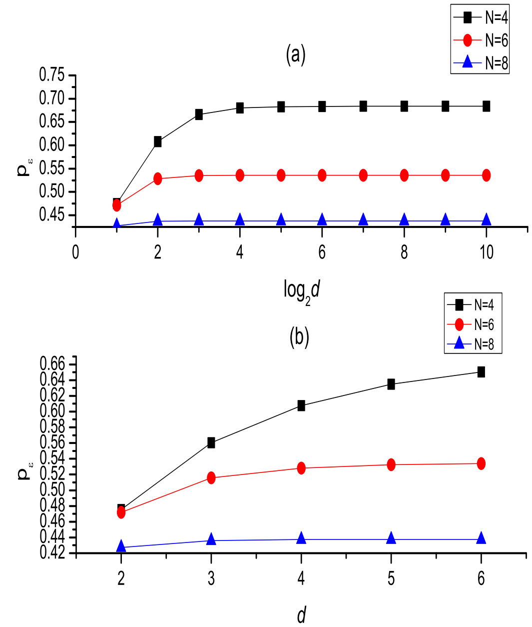

FIG. 2: (color online) For the depolarizing channel, we show the relation of p_ϵ with N and d. Here we choose $\alpha_i = 1/\sqrt{d}$ for convenience, $\epsilon = 0.01$ and $N = 4$ (black cubic), $N = 6$ (red circle) and $N = 8$ (blue triangle) respectively. (a): the behavior of p_ϵ for large d. (b): the behavior of p_ϵ for small d. It can be seen that when d is fixed, p_ϵ decreases with N while when N is fixed, it grows with d to a saturated value $1 - \epsilon^{1/N}$.

grows (FIG 1). Moreover, when N is big enough, considering p_ϵ^{ij} is very small, the first term in the LHS of Eq.(3) can be omitted, leading to $p_\epsilon \sim -(1/N)\ln\epsilon$ (FIG.2). For the phase damping channel, no matter what N is, no ESD happens. However, $p_\epsilon \sim -(1/N)\ln\epsilon$ still holds. We can see that so long as d is fixed, the scaling relation between p_ϵ and N is always the same with that in Ref.[8], where $d = 2$.

Then we fix N to study the relation between the entanglement robustness and d. For the depolarizing channel, from Eqs.(2) and (3) one can find that when the coefficients α_is are given (it's not necessary to set all α_is the same), both $p_{N/2}^{ij}$ (also $p_{N/2}$) and p_ϵ^{ij} (also p_ϵ) grow with d, meaning the entanglement is more robust (FIG.1 and 2). Moreover, when d is large enough, we have $\lim_{d\to\infty} p_{N/2} = 1$ (FIG.1) and $\lim_{d\to\infty} p_\epsilon = 1-\epsilon^{1/N}$ (FIG.2). For the phase damping channel, when N is fixed, it's obvious that both the ESD time (infinity) and $p_\epsilon^{ij} = 1 - \epsilon^{1/N}$ are independent of d.

One question arises that for the depolarizing channel, what's the dynamical behavior of the bipartite entanglement corresponding to the least balanced partition that vanishes first? As we know, in qubit case (d=2) it vanishes earlier when N grows. Now our numerical calculation shows its relation with N and d (FIG.3).

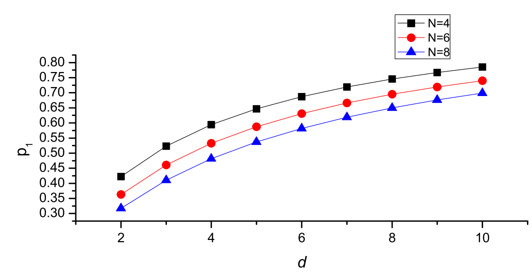

FIG. 3: (color online) For the depolarizing channel, we show the relation of p_1 with d and N. Here we choose $\alpha_i = 1/\sqrt{d}$ for convenience and $N = 4$ (black cubic), $N = 6$ (red circle) and $N = 8$ (blue triangle) respectively. It can be seen that when d is fixed, p_1 decreases with N while when N is fixed, it grows with d.

One could expect that the dependence of entanglement robustness on N and d would be similar, since in both cases we are increasing the dimension of the Hilbert space of each of the bi-partitions. But in fact their influences on the entanglement robustness are different. Roughly speaking, entanglement increases with d and thus it is more robust, while entanglement becomes more fragile with N since more particle are entangled together and it becomes easier to be destroyed. We take the depolarizing channel as an example, for which according to the discussion above, p_ϵ increases with d while it decreases with N. This can be explained as follows. If d is fixed, when we increase N, we increase the components of the state. Considering the depolarizing channel acts locally on every component, the growth of N will make the entanglement more fragile. In the depolarizing channel $\mathcal{E}_1(\rho) = (1-p)\rho + \frac{p}{d}\mathbf{1}$, p can be regarded as the probability with which ρ is broken by the channel. The probability of the N-particle state to be broken by the collective local action of N channels on each particle must grow with N (about Np as a rough estimation), leading that the entanglement becomes less robust. The probability with which the N-particle state is not broken can be estimated roughly as $(1-p)^N$ and this probability can also be expressed as ϵ considering the entanglement decay. Therefore we have $(1-p)^N = \epsilon$, leading to our familiar results. If N is fixed, with the increase of d, the influence of the channel on ρ will be more and more negligible, which leads the entanglement becomes robust.

V. SUMMARY

In this brief report, we mainly investigate the dynamical property of entanglement of a generalized N-qudit GHZ state under the influence of the depolarizing channel and the phase damping channel. We study the relation of the entanglement robustness with N and d. First we consider the ESD time t_1 and t_2 respectively of the bipartite entanglement corresponding to the most balanced partition (t_1) and the bipartite entanglement corresponding to the least balanced partition (t_2). When d is fixed, for the depolarizing channel t_1 delays when N grows whereas t_2 becomes earlier. For the phase damping channel, no ESD happens for any N. These results are qualitatively the same with that of $d = 2$. When N is fixed, for the depolarizing channel both t_1 and t_2 grow with d whereas still no ESD happens for the phase damping channel for any d. Next we consider the time at which the bipartite entanglement becomes arbitrarily small (t_3). When d is fixed the scaling relation between p_ϵ and N is totally independent of d therefore the same with the result in Ref.[8] for both channels. When N is fixed, for the depolarizing channel t_3 grows with d whereas it's independent of d for the phase damping channel. There are many other multiqudit entangled states such as generalized N-qudit W-state [9] and other quantum channels. It is not clear how the behaviors of the entanglement differ in those situations. This is worth being studied further.

VI. ACKNOWLEDGEMENTS

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